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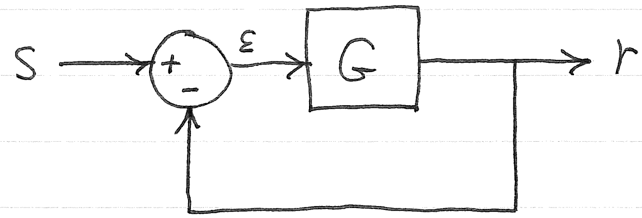
Feedback



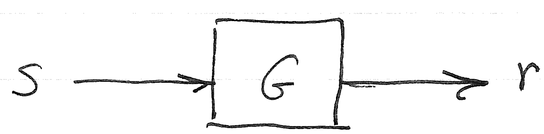
$$r = G\varepsilon = G(s - r)$$

$$(1+G)\,r = G s$$

$$r = \frac{G}{1+G}\, s$$

For large G, $\quad r \approx s$

But what's the big deal? We could get that result just by the system



$$r = G s$$

with $G \approx 1$

The big deal is not only that r is controlled by s but that it is insensitive to other things.
What "other things"?

One is G itself. In the feedback case

$$r = \frac{G}{1+G} s$$

We have

$$g = \frac{r}{s} = \frac{G}{1+G} = 1 - \frac{1}{1+G}$$

$$\frac{dg}{dG} = \frac{1}{(1+G)^2}$$

which can be made arbitrarily small by making G large

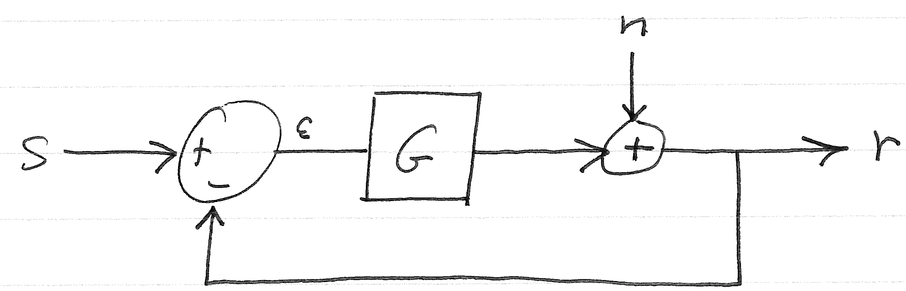
But in the non-feedback case

$$g = \frac{r}{s} = G$$

So $\frac{dg}{dG} = 1$

and there is no protection against changes in G.

Another example is some extraneous influence that tends to perturb the response, which we might call "noise".



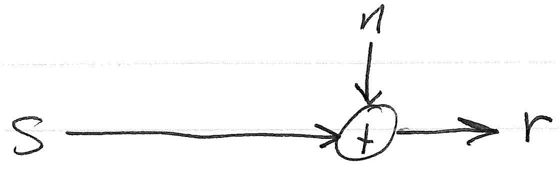
$$r = n + G(s - r)$$

$$(1+G)r = n + Gs$$

$$r = \left(\frac{1}{1+G}\right)n + \frac{G}{1+G}s$$

As $G \to \infty$, $r \to s$, independent of n

Compare this to the non-feedback system

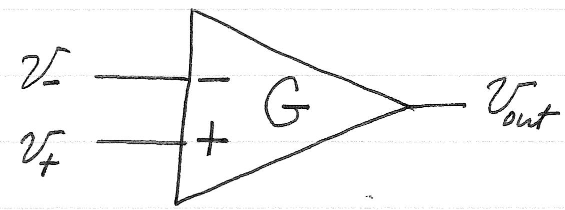


$$r = s + n$$

in which r is as sensitive to n as to s.

Physical Example: Operational Amplifier

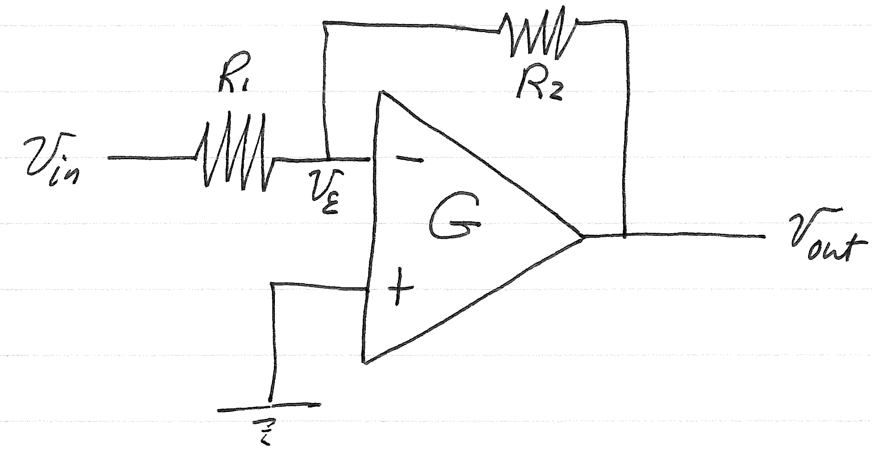
It is easy to build a differential amplifier with high gain and high input impedance



$$v_{out} = G(v_+ - v_-)$$

$G \approx 10^6$ but might vary from 10^5 to 10^7

How can we use this to make an amplifier of modest but reliable gain?



$$v_{out} = -G\, v_\varepsilon \qquad \frac{v_{in} - v_\varepsilon}{R_1} = \frac{v_\varepsilon - v_{out}}{R_2}$$

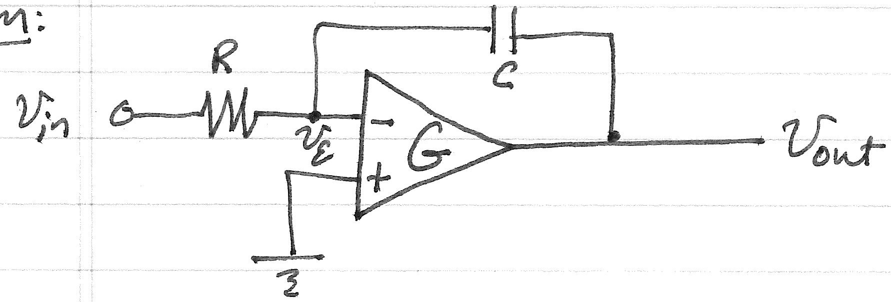
(result of high input impedance)

As $G \to \infty$, $v_\varepsilon \to 0$ and

$$\frac{v_{out}}{v_{in}} \to -\left(\frac{R_2}{R_1}\right)$$

Some other op-amp circuits analyzed by principle of virtual ground

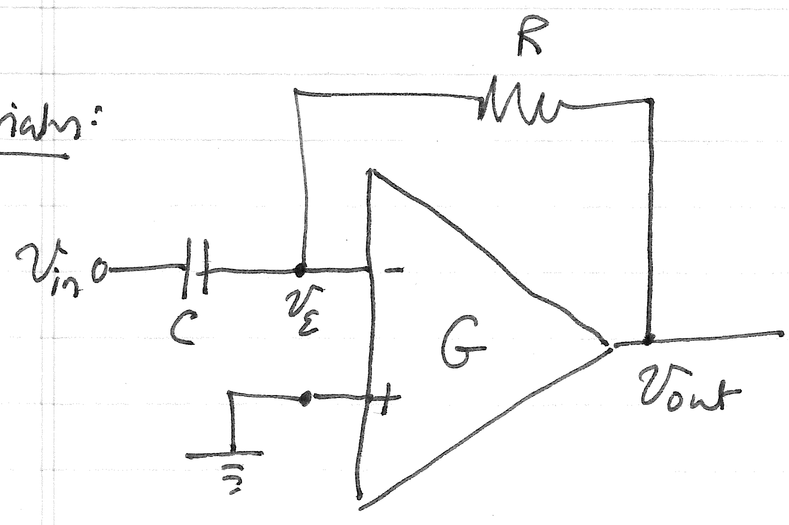
integrator:



In the limit $G \to \infty$, $v_\varepsilon \to 0$ and

$$-\frac{v_{in}}{R} = C\frac{dv_{out}}{dt} \Rightarrow v_{out} = -\frac{1}{RC}\int^{t} v_{in}\,dt'$$

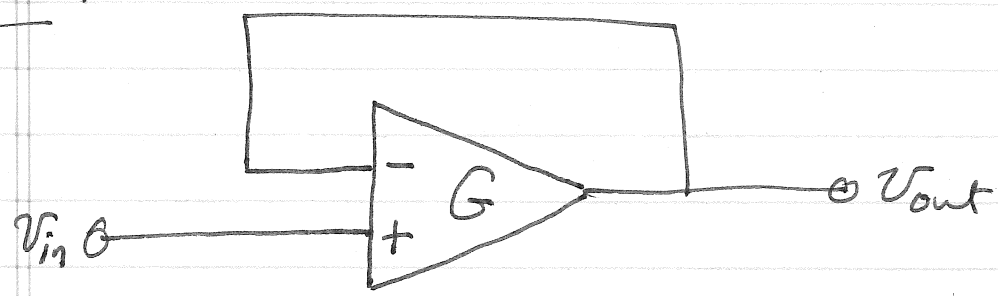
differentiator:



In the limit $G \to \infty$, $v_\varepsilon \to 0$ and

$$C\frac{dv_{in}}{dt} = -\frac{v_{out}}{R} \Rightarrow v_{out} = -(RC)\frac{dv_{in}}{dt}$$

follower:



$$v_{out} = G(v_{in} - v_{out})$$

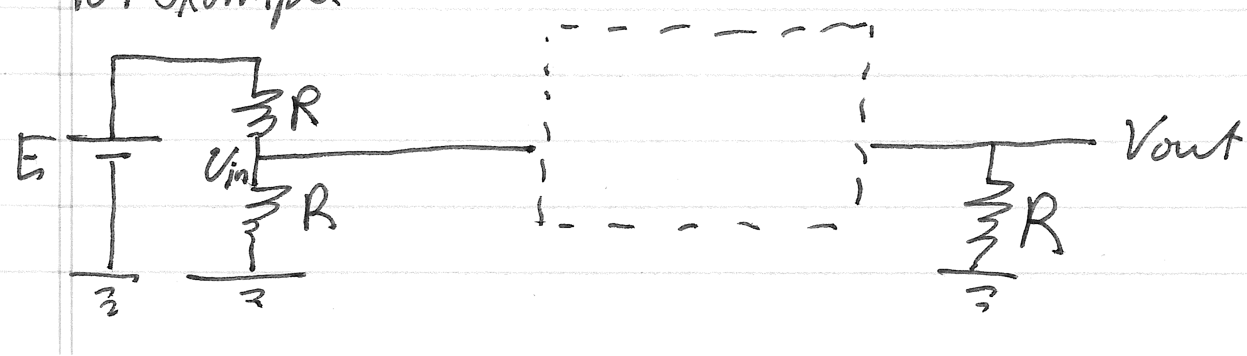
$$v_{out} = \frac{G}{1+G} v_{in} \rightarrow v_{in} \text{ as } G \rightarrow \infty.$$

Why is this better than:

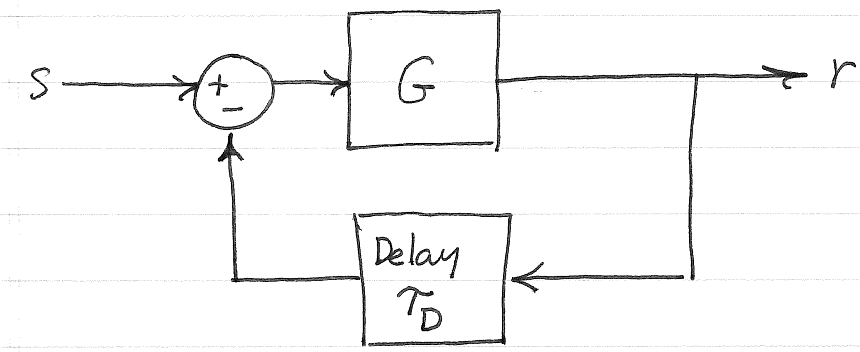
v_{in} o———————o v_{out} ?

Answer: Because we can connect anything to v_{out} (within reason) without affecting v_{in}.

For example



What about dynamics? Real systems always have <u>delays</u>, so consider



$$r(t) = G\left(s(t) - r(t-\tau_D)\right)$$

Consider, for example $s(t) = \begin{cases} 0 & \text{for } t<0 \\ 1 & \text{for } t>0 \end{cases}$

with $r(t)=0$ for $t<0$

Clearly $r(t)$ will be piecewise constant

$$r(t) = R_j \qquad j\tau_D < t < (j+1)\tau_D$$

with $R_j = 0$ for $j<0$, $R_0 = G$

$$R_j = G(1-R_{j-1}) \quad \text{for } j \geq 0$$

To solve this, let R be defined by

$$R = G(1-R) \qquad \text{i.e. } R = \frac{G}{1+G}$$

and subtract to get

$$R_j - R = -G(R_{j-1} - R)$$

$$R_j - R = (-G)^j (R_0 - R)$$

$$R_j = \frac{G}{1+G} + (-G)^j \left(G - \frac{G}{1+G}\right)$$

$$= \frac{G}{1+G} + (-G)^j G\left(1 - \frac{1}{1+G}\right)$$

$$= \frac{G}{1+G}\left(1 + (-G)^j G\right)$$

$$= \frac{G}{1+G}\left(1 - (-G)^{j+1}\right)$$

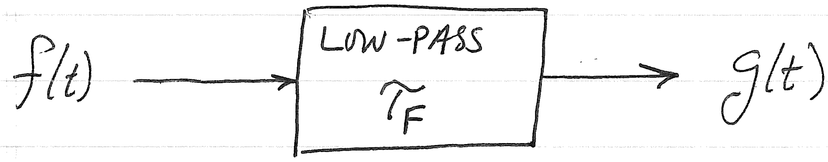
For $|G| < 1$, $R_j \to \frac{G}{1+G}$ as $j \to \infty$

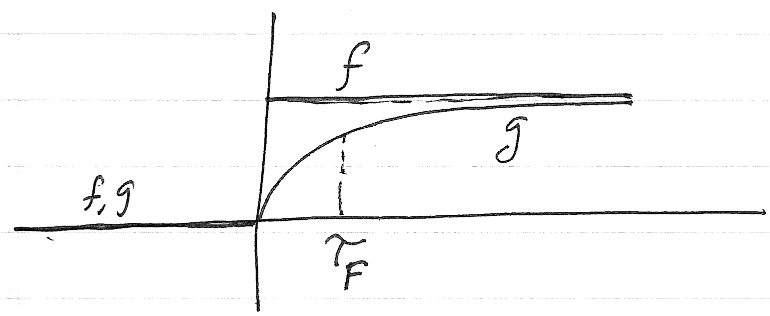
but for $|G| > 1$, $|R_j| \to \infty$ as $j \to \infty$

This is called instability. No matter how small the delay, the feedback system is unstable for $|G| > 1$.

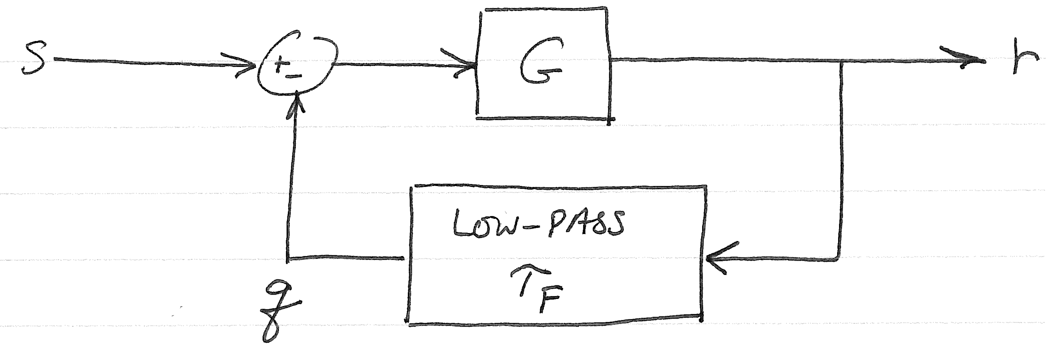
This seems like bad news: it seems to say that feedback with gain greater than 1 will never work in the real world.

Before giving up on feedback, let's try some other dynamics. Instead of a pure delay, let's try a "sloppy" delay, otherwise known as a "low-pass filter"





$$\boxed{\tau_F \frac{dg}{dt} + g = f}$$



$$r(t) = G\big(s(t) - q(t)\big)$$

$$q(t) + \tau_F \frac{dq}{dt} = r(t) = G\big(s(t) - q(t)\big)$$

$$(1+G)\, q(t) + \tau_F \frac{dq}{dt} = G\, s(t)$$

$$q(t) + \left(\frac{\tau_F}{1+G}\right)\frac{dq}{dt} = \frac{G}{1+G}\, s(t)$$

Suppose $s(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$

and $q(t) = r(t) = 0$ for $t < 0$

Then it is easy to check that

$$q(t) = \frac{G}{1+G}\left(1 - \exp\left(-(1+G)\frac{t}{\tau_F}\right)\right), \quad t>0$$

and therefore

$$r(t) = G\big(S(t) - g(t)\big)$$

$$= G\left(1 - \frac{G}{1+G}\left(1 - \exp\left(-(1+G)\frac{t}{T_F}\right)\right)\right)$$

$$= \frac{G}{1+G} + \frac{G^2}{1+G}\exp\left(-(1+G)\frac{t}{T_F}\right)$$

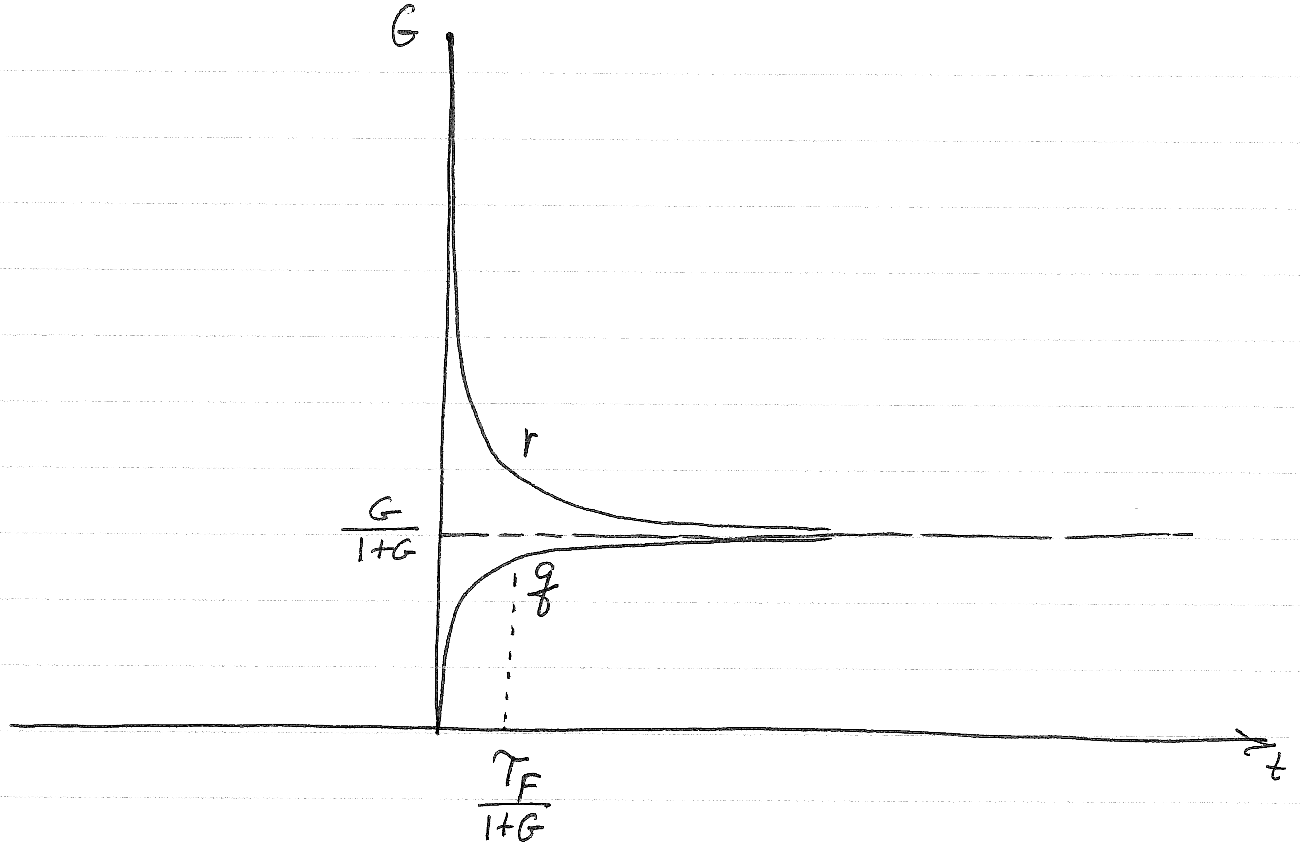
$$= \frac{G}{1+G}\left(1 + G\exp\left(-(1+G)\frac{t}{T_F}\right)\right)$$

$$\longrightarrow \frac{G}{1+G} \quad \text{as } t \to \infty$$

which is <u>stable</u>, <u>no matter how big G is</u>

Also, for any fixed $t > 0$:

$$\lim_{G \to \infty} r(t) = 1$$



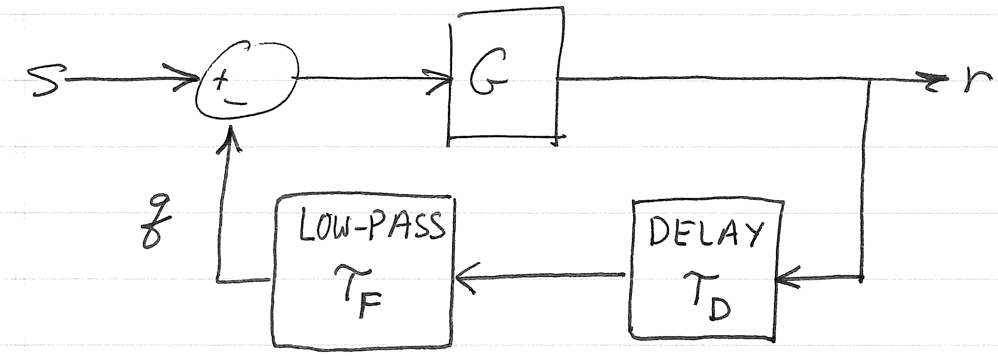
Summary

Pure delay → instability for all $G > 1$

Low pass filter → stability for all G

Real systems will have combination of pure delays and sloppy delays. Consider that case:

Feedback with pure delay and low-pass filter in the feedback path



$$q(t) + \tau_F \frac{dq}{dt}(t) = r(t-\tau_D) = G\Big(s(t-\tau_D) - q(t-\tau_D)\Big)$$

Note: setting $\tau_F = 0$ or $\tau_D = 0$ we recover the cases considered previously.

Instead of considering step response, we need a more general method for analyzing stability. That method is as follows:

Set $s(t) \equiv 0$ and look for solution of the form

$$q(t) = e^{i\omega t}$$

If such a solution exists with $\mathrm{Im}(\omega) \leq 0$, we say the system is "unstable". If not, it is stable. Note that neutral stability is here classified as instability.

If $s(t) \equiv 0$ and $g(t) = e^{i\omega t}$, then

$$1 + i\omega\tau_F = -G e^{-i\omega\tau_D}$$

We write this in the form

$$-1 = \frac{G e^{-i\omega\tau_D}}{1 + i\omega\tau_F}$$

Since we are interested in solutions with $\mathrm{Im}(\omega) \leq 0$ let

$$\omega = \xi - i\eta \quad , \quad \xi, \eta \text{ real} , \quad \eta \geq 0$$

$$-1 = \frac{G e^{-i\xi\tau_D} e^{-\eta\tau_D}}{1 + \eta\tau_F + i\xi\tau_F}$$

Separate this into amplitude and phase equations:

$$\frac{G e^{-\eta\tau_D}}{\sqrt{(1+\eta\tau_F)^2 + (\xi\tau_F)^2}} = 1$$

$$\xi\tau_D + \arctan\left(\frac{\xi\tau_F}{1+\eta\tau_F}\right) = k\pi \quad , \quad k \text{ an odd integer}$$

τ_D, τ_F, G are positive real constants. Unknowns are ξ, η which are real, and moreover $\eta \geq 0$.

For each η, left hand side of the phase equation is a continuous strictly increasing function of ξ which is unbounded as $\xi \to \pm\infty$. Therefore the phase equation has a unique solution ξ for each k, η, which we denote

$$\xi_k(\eta)$$

Thus, $\xi_k(\eta)$ is implicitly defined by

$$\xi_k(\eta)\tau_D + \arctan\left(\frac{\xi_k(\eta)\tau_F}{1+\eta\tau_F}\right) = k\pi$$

Note that $\xi_{-k}(\eta) = -\xi_k(\eta)$

Since only ξ^2 appears in the magnitude equation, we may restrict consideration to positive k. (Recall that k is required to be odd.)

Our problem is reduced to

$$f_k(\eta) = \frac{G e^{-\eta\tau_D}}{\sqrt{(1+\eta\tau_F)^2 + (\xi_k(\eta)\tau_F)^2}} = 1$$

By inspection

$$\xi_k(\eta) > 0 \quad \text{for } k > 0$$

$$\xi_1(\eta) < \xi_3(\eta) < \xi_5(\eta) < \cdots$$

and by implicit differentiation

$$\frac{d\xi_k}{d\eta} > 0 \quad \text{for } k > 0$$

So $f_k(\eta)$ is a continuous decreasing function

Moreover, $f_k(\eta) \to 0$ as $\eta \to \infty$

Therefore $f_k(\eta) = 1$ has a solution for $\eta \geq 0$ iff

$$f_k(0) = \frac{G}{\sqrt{1 + (\xi_k(0)\tau_F)^2}} \geq 1$$

That is, iff

$$G \geq \sqrt{1 + (\xi_k(0)\tau_F)^2} \qquad (*)$$

We have instability iff (*) holds for <u>any</u> k and stability iff (*) is false for <u>every</u> k.

So we only need to consider the smallest of the $\xi_k(0)$, namely $\xi_1(0)$. Thus, the critical value of G is

$$G_* = \sqrt{1 + (\xi_1(0)\,\tau_F)^2}$$

and we have instability for $G \geq G_*$ and stability for $G < G_*$.

The equation that determines $\xi_1(0)$ is

$$\xi_1(0)\,\tau_D + \arctan\left(\xi_1(0)\,\tau_F\right) = \pi$$

Let

$$\theta = \xi_1(0)\,\tau_D$$

Then

$$G_* = \sqrt{1 + \left(\theta\,\frac{\tau_F}{\tau_D}\right)^2}$$

where

$$\theta + \arctan\left(\theta\,\frac{\tau_F}{\tau_D}\right) = \pi$$

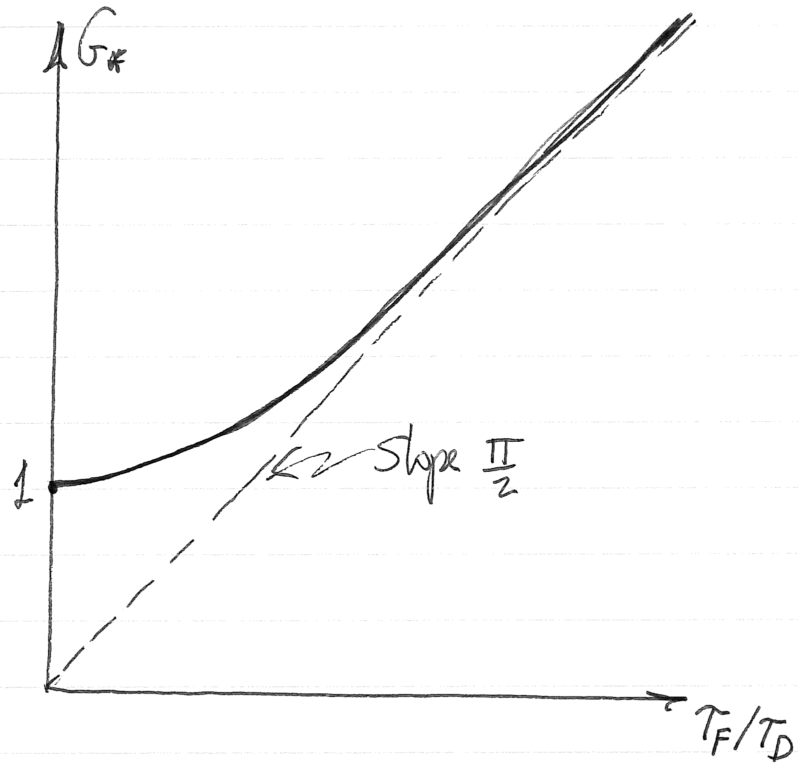
Thus G_* is a funct. of $\frac{\tau_F}{\tau_D}$

For $\frac{\tau_F}{\tau_D} \ll 1$, $\theta \approx \pi$, and

$$G_* \approx 1 + \frac{1}{2}\left(\pi \frac{\tau_F}{\tau_D}\right)^2$$

For $\frac{\tau_F}{\tau_D} \gg 1$, $\theta \approx \frac{\pi}{2}$

$$G_* \sim \frac{\pi}{2} \frac{\tau_F}{\tau_D}$$



Homework

1) Plot G_* as a function of τ_F/τ_D

Suggestion: Use θ as a parameter over the interval $\pi > \theta > \pi/2$

2) Simulate the feedback system with delay τ_D and low-pass filter τ_F in the case $\tau_F/\tau_D = 10$, for input given by a unit step

$$s(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

assuming that

$$g(t) = r(t) = 0 \quad \text{for } t < 0.$$

Do this in 3 cases: $G = \frac{1}{2} G_*$, $G = G_*$, $G = 2G_*$

Suggestion: For convenience choose $\Delta t = \tau_D/m$, where m is an integer. Then

$$g(t) + \tau_F \frac{g(t+\Delta t) - g(t)}{\Delta t} = G\big(s(t-\tau_D) - g(t-\tau_D)\big)$$

$$g(t+\Delta t) = \left(1 - \frac{\Delta t}{\tau_F}\right) g(t) + \frac{\Delta t}{\tau_F} G\big(s(t-\tau_D) - g(t-\tau_D)\big)$$

$$r(t) = G\big(s(t) - g(t)\big)$$

Be sure to choose Δt small enough that results don't vary much as Δt is further reduced.