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Gyrocompass

Let the center of mass of a gyroscope rotate with the earth, and let the axis of the gyroscope be constrained to lie in a locally horizontal plane.

Let $\underline{z}$ be a unit vector aligned with the earth's axis, pointing north, so that

(1) $\Omega \underline{z}$ = angular velocity of the earth.

At the location of the gyroscope, let

(2) $\{\underline{e}(t), \underline{n}(t), \underline{r}(t)\}$ = orthonormal triad of vectors pointing east, north, and up, respectively.

Note that

(3) $\underline{z} \cdot \underline{e}(t) = 0, \quad \underline{z} \cdot \underline{n}(t) = \cos\theta, \quad \underline{z} \cdot \underline{r}(t) = \sin\theta,$

where θ is the latitude of the gyroscope.

Since $\underline{e}(t)$, $\underline{n}(t)$, $\underline{r}(t)$ are rigidly attached to the rotating earth, they all satisfy the same differential equation

(4) $$\frac{d\underline{v}}{dt} = \Omega\, \underline{z} \times \underline{v} \quad ,$$

where $\underline{v}$ denotes any one of the vectors $\underline{e}, \underline{n}, \underline{r}$. Then, if $\underline{w}$ is also any one of these vectors, we have

(5) $$\frac{d\underline{v}}{dt} \cdot \underline{w} = \Omega (\underline{z} \times \underline{v}) \cdot \underline{w}$$

$$= \Omega\, \underline{z} \cdot (\underline{v} \times \underline{w}) \quad .$$

It follows that

$$(6) \quad \frac{d}{dt}\begin{pmatrix} \underline{e} \\ \underline{n} \\ \underline{r} \end{pmatrix} = \Omega \begin{pmatrix} 0 & \underline{z}\cdot\underline{r} & -\underline{z}\cdot\underline{n} \\ -\underline{z}\cdot\underline{r} & 0 & \underline{z}\cdot\underline{e} \\ \underline{z}\cdot\underline{n} & -\underline{z}\cdot\underline{e} & 0 \end{pmatrix}\begin{pmatrix} \underline{e} \\ \underline{n} \\ \underline{r} \end{pmatrix}$$

$$= \Omega \begin{pmatrix} 0 & \sin\theta & -\cos\theta \\ -\sin\theta & 0 & 0 \\ \cos\theta & 0 & 0 \end{pmatrix}\begin{pmatrix} \underline{e} \\ \underline{n} \\ \underline{r} \end{pmatrix}$$

To get the first line of the foregoing, we have used $\underline{e}\times\underline{n}=\underline{r}$, $\underline{n}\times\underline{r}=\underline{e}$, $\underline{r}\times\underline{e}=\underline{n}$. To get the second line, we used equation (3).

Two more unit vectors that will be useful in the following are

(7) $\underline{a}(t) =$ a vector aligned with the axis of the gyroscope

and

(8) $$\underline{b}(t) = \underline{r}(t) \times \underline{a}(t)$$

Since axis is constrained to lie in a locally horizontal plane, we may write

(9) $$\underline{a}(t) = \cos(\phi(t))\,\underline{e}(t) + \sin(\phi(t))\,\underline{n}(t)$$

(10) $$\underline{b}(t) = -\sin(\phi(t))\,\underline{e}(t) + \cos(\phi(t))\,\underline{n}(t)$$

Our goal is to derive an equation of motion for the angle $\phi(t)$.

Let $\underline{\omega}(t)$ be the angular velocity of the gyroscope. We can write it in the basis $\{\underline{a}(t), \underline{b}(t), \underline{r}(t)\}$ as

(11) $$\underline{\omega}(t) = \omega_a(t)\,\underline{a}(t) + \omega_b(t)\,\underline{b}(t) + \omega_r(t)\,\underline{r}(t)$$

Since the vector $\underline{a}$ moves rigidly with the material of the rotating gyroscope, it has the differential equation

(12) $$\frac{d\underline{a}}{dt} = \underline{\omega}(t) \times \underline{a}(t) = \omega_r(t)\,\underline{b}(t) - \omega_b(t)\,\underline{r}(t)$$

Since $\underline{a}(t)\cdot\underline{r}(t) = 0$, this determines $\omega_b(t)$ as follows

~~(12)~~ (12.5) $$\omega_b(t) = -\frac{d\underline{a}}{dt}\cdot\underline{r}(t) = \underline{a}(t)\cdot\frac{d\underline{r}}{dt}$$

$$= \underline{a}(t)\cdot \Omega\cos\theta\;\underline{e}(t)$$

$$= \Omega\cos(\theta)\,\cos\phi(t)$$

We assume that the mass distribution of the gyroscope is axisymmetric. This implies that the angular momentum of the gyroscope is given by

(13) $$\underline{L}(t) = I_0 \omega_a(t)\, \underline{a}(t) + I_1 \left(\omega_b(t)\, \underline{b}(t) + \omega_r(t)\, \underline{r}(t) \right)$$

where I_0 is the moment of inertia about the symmetry axis, and I_1 is the moment of inertia about any axis perpendicular to the symmetry axis.

The gyroscope is mounted in a circular frame that is rigidly attached to the rotating earth. The frame lies in a locally horizontal plane. The gyroscope makes contact with the frame only at the two points where the axis of the gyroscope intersects the circular frame. Thus the torque applied by the frame to the gyroscope must be of the form

(14) $$\underline{\tau}(t) = \tau_b(t)\,\underline{b}(t) + \tau_r(t)\,\underline{r}(t)$$

since there cannot be any torque about the axis $\underline{a}(t)$. (We are assuming here that a frictionless bearing allows free rotation of the gyroscope about its axis.)

The torque $\tau_b(t)$ is a constraint torque. It takes on whatever value is needed to keep the axis of the gyroscope in the locally horizontal plane.

The torque $\tau_r(t)$ is a frictional torque that resists rotation of the axis $\underline{a}(t)$ relative to the circular frame that is attached to the earth. We therefore assume that $\tau_r(t)$ is given by

(15) $$\tau_r(t) = -\gamma \frac{d\phi}{dt}$$

This assumes there is oil in the track so that the frictional ~~to~~ force arises from motion in a viscous fluid.

The equation of motion for $\underline{L}(t)$ is

(16) $$\frac{d\underline{L}}{dt} = \tau_b(t)\,\underline{b}(t) + \tau_r(t)\,\underline{r}(t)$$

Since the right-hand side is already expressed in terms of the basis $\{\underline{a}(t), \underline{b}(t), \underline{r}(t)\}$, our task now is to express the left-hand side, $d\underline{L}/dt$, in terms of this basis.

From (13),

(17) $$\frac{d\underline{L}}{dt} = I_0 \frac{d\omega_a}{dt}\,\underline{a}(t) + I_1\left(\frac{d\omega_b}{dt}\,\underline{b}(t) + \frac{d\omega_r}{dt}\,\underline{r}(t)\right) + I_0\,\omega_a(t)\,\frac{d\underline{a}}{dt} + I_1\left(\omega_b(t)\,\frac{d\underline{b}}{dt} + \omega_r(t)\,\frac{d\underline{r}}{dt}\right)$$

Therefore,

(18) $$\underline{a}(t)\cdot\frac{dL}{dt} = I_0\frac{dW_a}{dt} + I_1\left(\omega_b(t)\,\underline{a}(t)\cdot\frac{d\underline{b}}{dt} + \omega_r(t)\,\underline{a}(t)\cdot\frac{d\underline{r}}{dt}\right)$$

But

(19) $$\underline{a}(t)\cdot\frac{d\underline{b}}{dt} = -\frac{d\underline{a}}{dt}\cdot\underline{b}(t) = -\omega_r(t)$$

(20) $$\underline{a}(t)\cdot\frac{d\underline{r}}{dt} = -\frac{d\underline{a}}{dt}\cdot\underline{r}(t) = +\omega_b(t)$$

see (12). Thus, the coefficient of I_1 on the right-hand side of (18) is zero, and (18) becomes

(21) $$\underline{a}(t)\cdot\frac{dL}{dt} = I_0\frac{dW_a}{dt}$$

From (16) & (21) we reach the important conclusion that ω_a is a constant of the motion.

By applying $\underline{b}(t)\cdot$ to both sides of (16) and making use of (17) we get

(22) $$\tau_b(t) = \underline{b}(t)\cdot\frac{d\underline{L}}{dt} =$$

$$I_1\frac{d\omega_b}{dt} + I_0\,\omega_a\,\frac{d\underline{a}}{dt}\cdot\underline{b}(t) + I_1\,\omega_r(t)\frac{d\underline{r}}{dt}\cdot\underline{b}(t)$$

Each of the terms on the right-hand side of (22) can be evaluated. From (12),

(23) $$\frac{d\omega_b}{dt} = -\Omega\cos(\theta)\sin(\phi(t))\,\frac{d\phi}{dt}$$

From (12),

(24) $$\frac{d\underline{a}}{dt}\cdot\underline{b}(t) = \omega_r(t)$$

and from (6) & (10),

(25) $$\frac{d\underline{r}}{dt} \cdot \underline{b}(t) = \Omega \cos(\theta)\, \underline{e}(t) \cdot \underline{b}(t)$$

$$= -\Omega \cos(\theta) \sin(\phi(t))$$

Thus, (22) becomes

(26) $$\hat{\tau}_b(t) = -I_1 \Omega \cos(\theta) \sin(\phi(t)) \frac{d\phi}{dt}$$

$$+ I_0 \omega_a \omega_r(t) - I_1 \Omega \cos(\theta) \sin(\phi(t))\, \omega_r(t)$$

$$= I_0 \omega_a \omega_r(t) - I_1 \Omega \cos(\theta) \sin(\phi(t)) \left(\omega_r(t) + \frac{d\phi}{dt}\right)$$

This is a formula for the constraint torque that is applied by circular frame to keep the axis of the gyroscope in a locally horizontal plane. This formula will not appear in our equation of motion, but it tells us what the constraint torque is, in case that may be of interest.

Finally, we apply $\underline{r}(t)\cdot$ to both sides of (16) and make use of (17). This gives

(27) $$\tau_r(t) = \underline{r}(t)\cdot\frac{d\underline{L}}{dt} =$$

$$I_1\frac{d\omega_r}{dt} + I_0\omega_a\frac{d\underline{a}}{dt}\cdot\underline{r}(t) + I_1\omega_b(t)\frac{d\underline{b}}{dt}(t)\cdot\underline{r}(t)$$

$$= I_1\frac{d\omega_r}{dt} - \left(I_0\omega_a\underline{a}(t) + I_1\omega_b(t)\underline{b}(t)\right)\cdot\frac{d\underline{r}}{dt}$$

From (6),

(28) $$\frac{d\underline{r}}{dt} = \Omega\cos(\theta)\,\underline{e}(t)$$

and from (9) & (10), $\underline{a}(t)\cdot\underline{e}(t) = \cos(\phi(t))$, $\underline{b}(t)\cdot\underline{e}(t) = -\sin\phi(t)$.

Also $\omega_b(t)$ is given by equation (12.5).

Thus, equation (27) becomes

$$(29)\quad \tau_r(t) = I_1\left(\frac{d\omega_r}{dt} + \Omega^2\cos^2\theta\,\sin(\phi(t))\cos(\phi(t))\right)$$

$$- I_0\,\omega_a\,\Omega\cos\theta\,\cos(\phi(t))$$

This will become our equation of motion for $\phi(t)$. We already have τ_r in terms of ϕ, see equation (15). We need to rewrite $d\omega_r/dt$ in terms of ϕ. To do so, we differentiate on equation (9) with respect to t and compare the result to equation (12). From (9),

$$(30)\quad \frac{d\underline{a}}{dt} = \frac{d\phi}{dt}\,\underline{b}(t) + \cos(\phi(t))\frac{d\underline{e}}{dt}$$

$$+ \sin(\phi(t))\frac{d\underline{n}}{dt}$$

From (12),

$$(31)\quad \omega_r(t) = \underline{b}(t)\cdot\frac{d\underline{a}}{dt}$$

$$= \frac{d\phi}{dt} + \cos(\phi(t))\,\underline{b}(t)\cdot\frac{d\underline{e}}{dt}$$

$$+ \sin(\phi(t))\,\underline{b}(t)\cdot\frac{d\underline{n}}{dt}$$

From (6) & (10),

(32) $$\underline{b}(t) \cdot \frac{d\underline{e}}{dt} = \Omega \sin\theta \cos(\phi(t))$$

(33) $$\underline{b}(t) \cdot \frac{d\underline{n}}{dt} = \Omega \sin\theta \sin(\phi(t))$$

Thus, (31) becomes

(34) $$\omega_r(t) = \frac{d\phi}{dt} + \Omega \sin\theta$$

and it follows that

(35) $$\frac{d\omega_r}{dt} = \frac{d^2\phi}{dt^2}$$

Substituting this and (15) into (29), we get, finally, the equation of motion for ϕ:

(36) $$0 = I_1 \frac{d^2\phi}{dt^2} + \gamma \frac{d\phi}{dt}$$

$$- \Omega \cos\theta \cos\phi(t) \left(I_0 \omega_a - I_1 \Omega \cos\theta \sin\phi(t) \right)$$

Since $|\omega_a| >> |\Omega|$, this is approximately the same as

(37) $$0 = I_1 \frac{d^2\phi}{dt^2} + \gamma \frac{d\phi}{dt} - I_0 \omega_a \Omega \cos\theta \cos\phi(t)$$

$$= I_1 \frac{d^2\phi}{dt^2} + \gamma \frac{d\phi}{dt} + U'(\phi)$$

where

(38) $$U(\phi) = - I_0 \omega_a \Omega \cos\theta \sin\phi(t)$$

Note that the approximation made above based on $|\omega_a| >> |\Omega|$ is uniformly valid with respect to lattitude θ and axis orientation ϕ of the gyroscope, since $\cos\theta$ and $\sin\phi(t)$ are bounded by 1 in absolute value and since I_0 and I_1 are the same order of magnitude.

For the earth, $\Omega > 0$, i.e., the earth rotates towards the east. Assuming now that $\omega_a > 0$, the function $U(\phi)$ has a unique ~~minimum~~ minimum at $\phi = \pi/2$, which means that the vector $\underline{a}$ is pointing north, see equation (9).

Thus, the axis of the gyroscope will settle down to an orientation in which the spin of the gyroscope and the spin of the earth are as nearly parallel as possible, given the lattitude of the gyroscope and the constraint that its axis must lie in a locally horizontal plane.

Once ϕ has settled down to the constant value $\pi/2$, the torque that keeps the axis of the gyroscope rotating with respect to an inertial frame so that it maintains a fixed orientation with respect to the earth is given by (26) with $d\phi/dt = 0$ and $\omega_r = \Omega \sin\theta$, see (34). Thus

(39) $$\tau_b = I_0 \omega_a \Omega \sin\theta - I_1 \Omega^2 \sin\theta \cos\theta$$

This is zero at the equator $(\theta = 0)$ since in that case the stable orientation of the gyroscope corresponds to an angular momentum parallel to that of the earth and constant in an inertial frame. At the poles $(\theta = \pm\pi/2)$ $\tau_b = \pm I_0 \omega_a \Omega$, and this is the

torque needed to keep the gyroscope in a fixed orientation with respect to the earth after friction has degraded the relative motion of the axis in the frame.

In the neighborhood of the stable orientation $\phi = \frac{\pi}{2}$, equation (37) reduces to

(40) $$0 = I_1 \frac{d^2\tilde{\phi}}{dt^2} + \gamma \frac{d\tilde{\phi}}{dt} + I_0 \omega_a \Omega (\cos\theta) \tilde{\phi}$$

where

(41) $$\tilde{\phi} = \phi - \pi/2$$

Equation (40) describes a damped harmonic oscillator. Since we want the gyrocompass to settle down as quickly as possible to its stable orientation, we should choose critical damping, in which the two roots of the characteristic polynomial of (40) are equal. This gives a formula for γ, namely

(42) $$\gamma = \sqrt{4 I_1 I_0 \omega_a \Omega \cos\theta}$$

Note that the optimal choice of γ is latitude-dependent.

With γ given by (42), the roots of the characteristic polynomial of (40) are both equal to $-\lambda$, where

(43) $$\lambda = \sqrt{\frac{I_0}{I_1}\,\omega_a \Omega \cos\theta}$$

This has units of 1/time, and it gives the rate of approach of the gyrocompass to its stable orientation in the case of critical damping. Apart from dimensionless factors, this is governed by

(44) $$\sqrt{\omega_a \Omega}$$

which is the geometric mean of the angular velocity of the gyrocompass and the angular velocity of the earth.